

A Leading Order (But More than One Loop) Calculation of Structure Functions

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Outline

DIS Scattering of electrons and protons

$$\frac{d^2\sigma}{dx dQ^2} = \frac{2\pi\alpha^2}{Q^4 x} \left(2 - 2y + \frac{y^2}{1+R} \right) F_2(x, Q^2)$$

$$R = \frac{F_L}{F_2 + F_L} \quad (R \leq 0.4)$$

$$y = \frac{Q^2}{xS} = \frac{Q^2}{90000 x} \leq 0.6$$

most points $y < 0.25$

∴ HERA measures $F_2(x, Q^2)$ at small x .

Sharp rise at small x ($1 \lesssim x \lesssim 10^{-3}$)

Theory

$$d\sigma_{had} = \sum_a \int d\hat{\sigma}_{part} f_a$$

$d\hat{\sigma}_{part}$ perturbative cross section

f_a basic parton distribution

$d\hat{\sigma}_{part}$ infrared divergent. Factorization theorem
 \rightarrow divergences factored and absorbed into
 non perturbative f_a .

$$F(x, Q^2) = \sum_a \int \frac{dz}{z} L_a \left(\frac{x}{z}, \frac{Q^2}{z u^2} \right) f_a(z, u^2) + O(\frac{1}{Q^2})$$

Physics independent of u_F (choice $u^2 = Q^2$)
 and method of factoring divergences.

$L_a \left(\frac{x}{z}, \frac{Q^2}{z u^2} \right)$ perturbative. Process dependent
 and factorization scheme (FS) dependent.

$f_a(z, u^2)$ universal. u_F dependence
 \rightarrow Altarelli-Parisi equation.

$$\frac{df_a(z, u^2)}{du^2} = \sum_b \int \frac{dz}{z} P_{ab}(z, u^2, z) f_b \left(\frac{z}{u^2}, u^2 \right)$$

$P_{ab}(z, u^2, z)$ perturbative but FS dependent.

Underlying $f_b(z, u^2)$ non perturbative

Usual \rightarrow Calculate C_F and P_{ab} order by order in $\alpha_s(x)$. Gauss $f_+(x, \bar{Q}^2)$ and calculate $F_{ab}(x)$ for $Q^2 > Q_0^2$. (NLO in α_s).

$$\text{But } P_{ab}(C_F) \rightarrow \frac{\partial^m}{\partial x^m} \ln \left(\frac{1}{x} \right) \quad m=0 \rightarrow m-1$$

$$\text{At HERA} \quad x < 10^{-4} \rightarrow \ln \left(\frac{1}{x} \right) > 9.$$

BFKL equation. \rightarrow Leading power in $\ln \left(\frac{1}{x} \right)$ for each power in α_s (for gluon).

$$\rightarrow F_2 \sim x^{-\lambda} \quad (\lambda = 4 \ln 2 \bar{L} \approx 0.5) \quad x \rightarrow 0.$$

This is too steep.

More detailed calculations using leading $\ln \left(\frac{1}{x} \right)$ C_F and P_{ab} (Cotani + Hautmann), seem to depend on factorization scheme.

Generally including leading $\ln \left(\frac{1}{x} \right)$ terms \rightarrow worse fits than NLO in $\alpha_s(\bar{Q}^2)$ calculations with

$$f_+(x, \bar{Q}^2) \sim x^{-0.27} \quad \text{at small } x.$$

$$\bar{Q}^2 \approx 2 \cdot 10^4$$

Aim of my Work

To provide (as far as possible) unambiguous calculations of structure functions valid for all x^2 (above 1 GeV^2) and x .

In particular answer questions.

Why does there seem to be factorization scheme dependence in calculations with leading $\alpha(\gamma_x)$ terms?

Is there any real prediction for inputs at small x ?

How does one include both leading $\alpha(\gamma_x)$ terms and leading $\alpha(\alpha')$ terms?

Is it necessary to include leading $\alpha(\gamma_x)$ terms any way?

Solutions \rightarrow

Work in moment space

$$F(n, \alpha^2) = \int_0^n dx x^{n+1} F(x, \alpha^2)$$

$$\gamma_{bf}(n, \alpha^2) \Rightarrow \int_0^n dx x^n P_{bf}(x, \alpha^2)$$

$$(F(n, \alpha^2) \sim F(n, \alpha^2), \quad \tilde{f}(n, \alpha^2) \sim \gamma(n, \alpha^2)).$$

$$\rightarrow \frac{d\tilde{f}_n}{d\alpha^2} = \sum_b \gamma_{bf}(n, \alpha^2) \tilde{f}_{bf}(n, \alpha^2)$$

$$F(n, \alpha^2) = \sum_b \gamma_{bf}(n, \alpha^2) \tilde{f}_{bf}(n, \alpha^2).$$

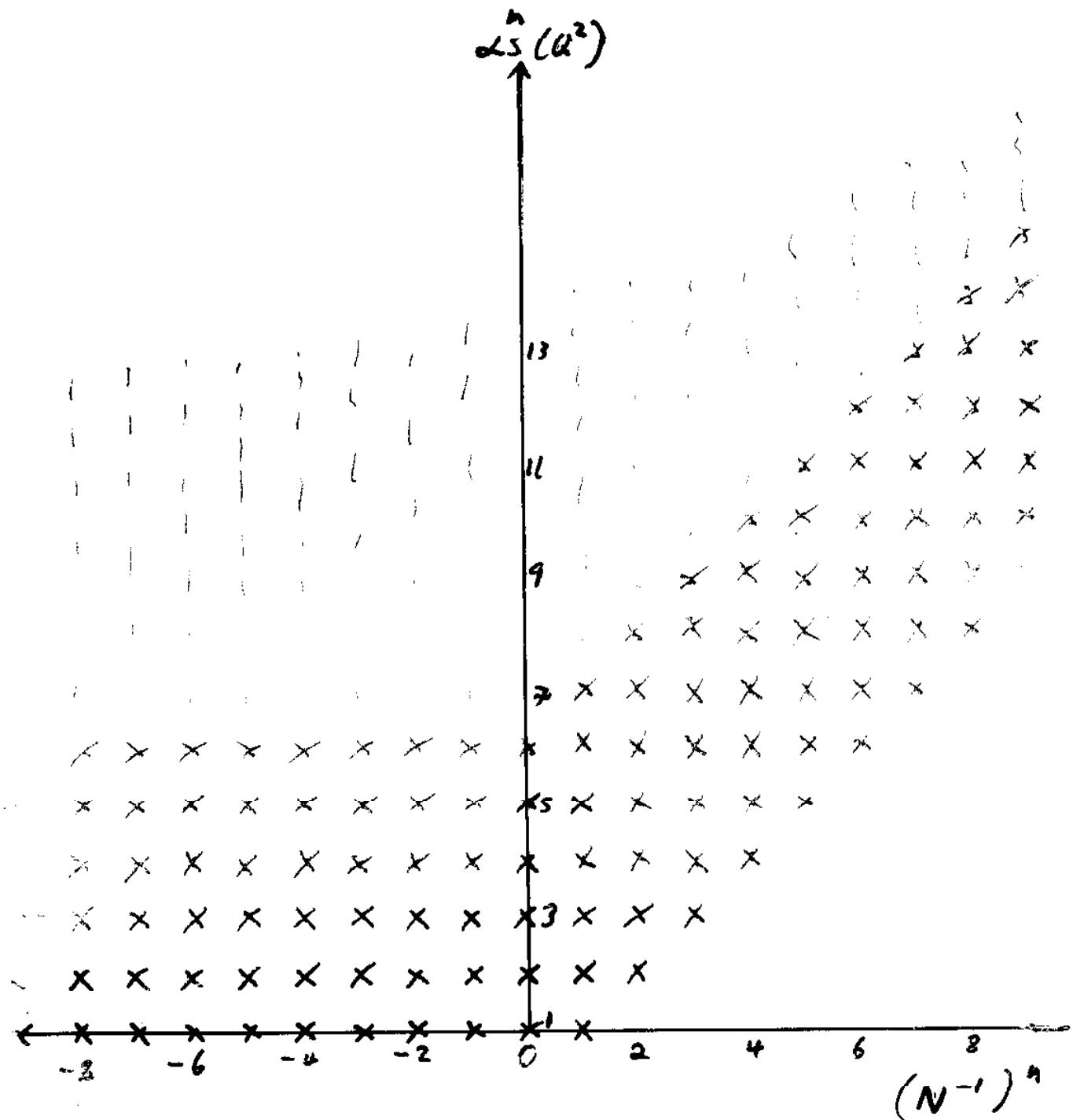
In moment space leading $\ln(\frac{1}{\alpha^2}) \rightarrow$ leading N^{-1}

$$\frac{\alpha^2}{(x)} (\alpha^2 \ln(\frac{1}{\alpha^2}))^{N-1} \rightarrow \left(\frac{\alpha^2}{N} \right)^N$$

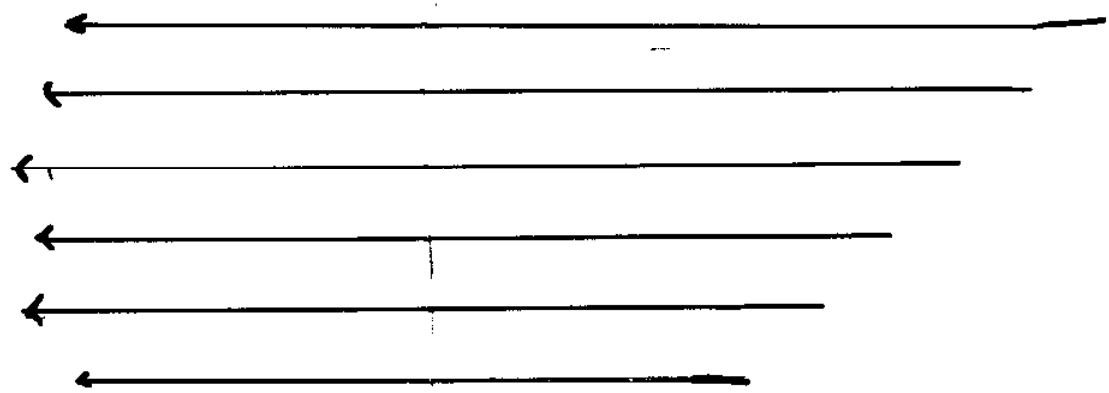
Leads to leading in $\ln(\frac{1}{\alpha^2})$

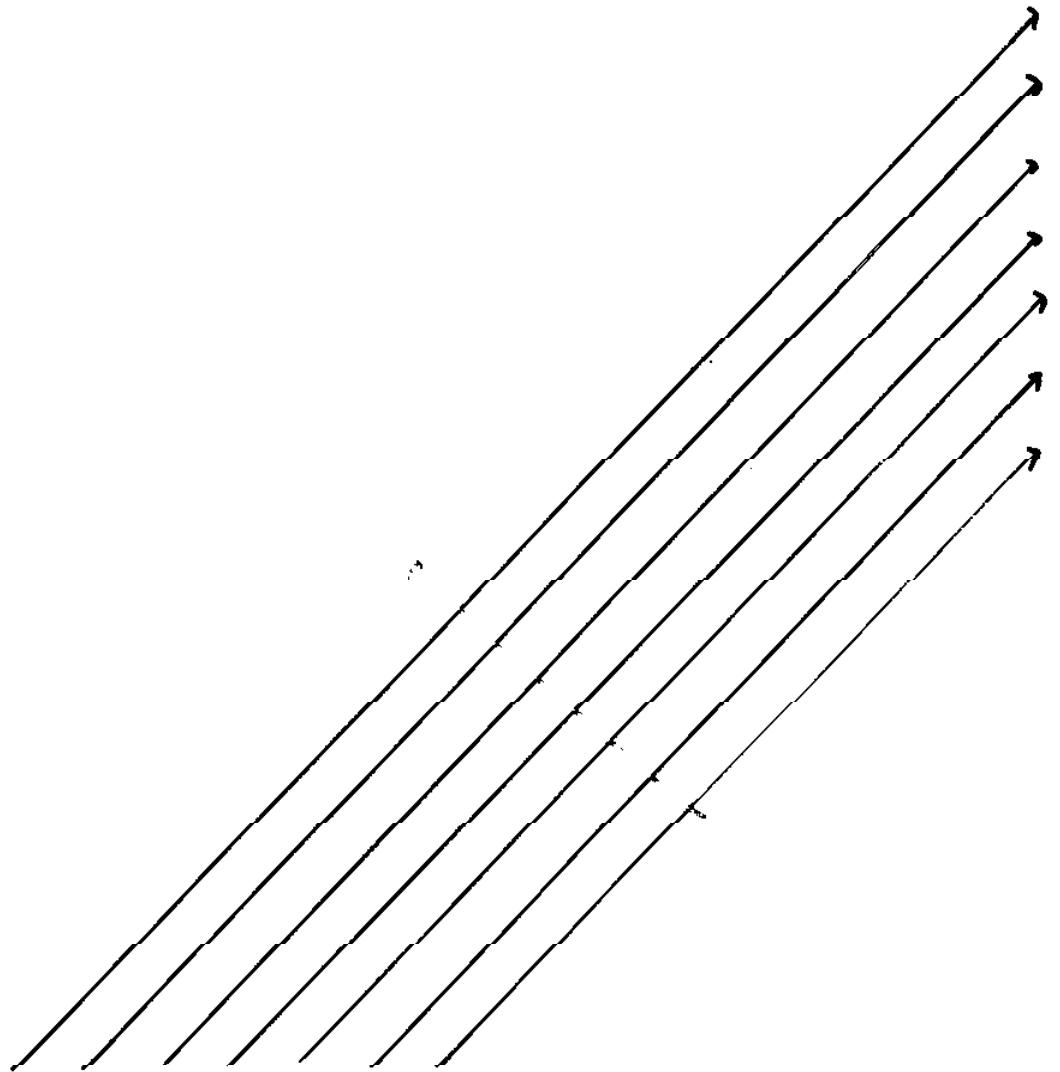
$$\gamma_{gg} (\gamma_{gf}) = \sum_{n=1}^{\infty} a_n \left(\frac{\alpha^2}{N} \right)^n$$

$$\gamma_{fg} (\gamma_{ff}, \zeta_i) = \alpha^2 \sum_{n=0}^{\infty} b_n \left(\frac{\alpha^2}{N} \right)^n$$



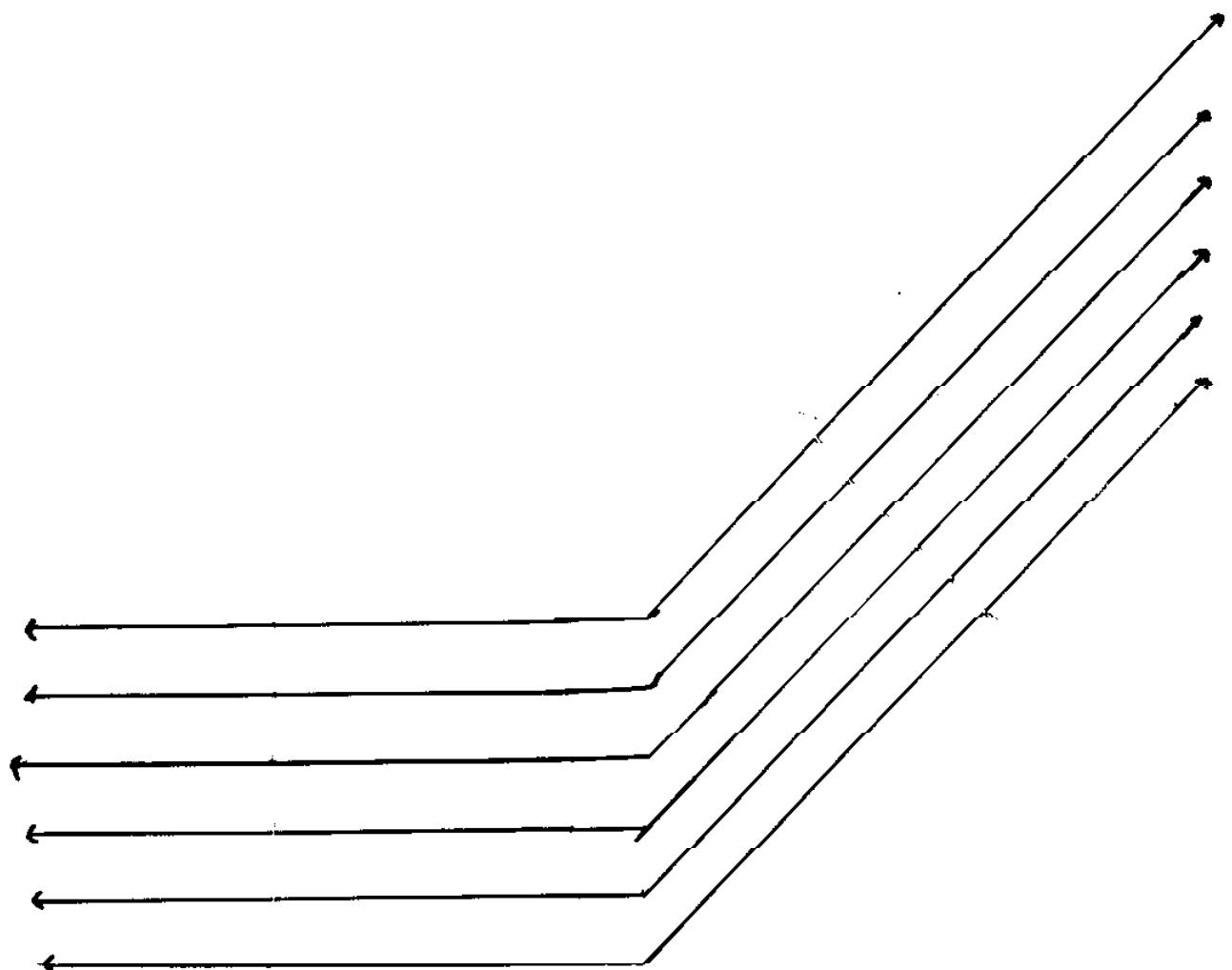
Expand in $\Delta S(Q^2)$ horizontal lines
 $\ln L'(x)$ diagonals
 correctly (i.e. both) envelopes.





$$\delta_{gg} \sim \frac{1}{N} - 4 + 12N -$$

$$\Gamma_{2L} \sim \frac{1}{N} - 1.5 + 3N -$$



Both f 's and χ 's (and parton distributions) are physical, F 's - dependent quantities.

→ Dangerous to attach too much significance to them.

Also dangerous to order calculations according to them (as is usually done in all expansion schemes).

$$\text{i.e. } F_n(Q^2) \neq \sum_{m=0}^n G_m(Q^2) \tilde{f}_m(Q^2).$$

Alternative - For structure functions expand inputs and evolution in well defined way.

$$F(Q^2) = F_2(Q^2) E(10^2, Q_2^2).$$

$$E(10^2) = \sum_{n=0}^{\infty} E_n(10^2) \quad (\text{Input})$$

$$E(10^2, Q_2^2) = \sum_{n=0}^{\infty} E_n(10^2, Q_2^2) \quad (\text{Evolution}).$$

$$F_n(Q^2) = \sum_{m=0}^{\infty} F_{nm}(Q_2^2) E_m(Q^2, Q_2^2).$$

Equivalent to normal expansion order by order in α_s
but not in $\ln(\frac{Q^2}{Q_0^2})$ expansion.
→ $\ln(1 + x)$ is a rare

Avoid problems by using

$$F^{L^0}(N, \alpha^2) = F_I^{L^0}(N, Q_I^2) E_{L^0}(N, \alpha^2; Q_I^2)$$

$$F^{N^0}(N, \alpha^2) = F^{L^0}(N, Q^2)$$

$$+ F_I^{L^0}(N, Q_I^2) E_{N^0}(N, Q^2; Q_I^2) + F_I^{N^0}(N, Q_I^2) E_{L^0}(N, Q^2; Q_I^2).$$

In any expansion scheme

"Physical Anomalous Dimensions" of Cabani naturally appear.

$$\frac{\delta F_L}{\delta l \cdot Q^2} = \Gamma_{22} F_2 + \Gamma_{2e} F_L$$

$$\frac{\delta F_L}{\delta l \cdot Q^2} = \Gamma_{22} F_2 + \Gamma_{ee} F_L$$

But methods not identical.

$$F_2(x, t^2, +) = F_2(N, Q^2) e^{\frac{t^2}{N} \ln \frac{Q^2}{Q_0^2}}$$

$$\approx L(Q^2) \sum_{n=0}^{\infty} f_n \left(\frac{\ln(Q^2)}{N} \right)^n$$

$$(g(x, Q^2) + f(x, Q^2)) \leq_0 L(Q^2) f(x, N).$$

f_S dependent

f_S dependent.

Absolutely no information

Invariance under arbitrary scale δx

$$F_2(x, Q^2) = L(Q^2) F_2(x) e^{\frac{Q^2 - Q_0^2}{N} \ln \left(\frac{Q^2}{Q_0^2} \right)}$$

$A \rightarrow$ non perturbative scale

Fund. \rightarrow flat non perturbative input (Regge Physics)

$Q^2 \rightarrow \gtrsim 10 \text{ GeV}^2$ ($L(Q^2)$ too large).

$\lesssim 100 \text{ GeV}^2$ ($\ln Q^2 A$ too large).

$$\rightarrow F_2(x, Q^2) \sim x^{-0.3} \quad 0.01 < x < 10^{-5}$$

$$\left(\frac{\partial F_2(N, Q^2, \tau)}{\partial \ln Q^2} \right)^{(0)} \approx \omega_s(Q^2) \Gamma_{22}' \left(\frac{\omega_s(Q^2)}{N} \right) F_2(N) + \\ \delta_{gg}^0 \left(\frac{\omega_s}{N} \right) b(Q^2/N) \int \delta_{gg}^0 dt / Q^2$$

$$\Gamma_{22}' = \frac{\delta_{fg}^0 + C_{21}^2 \delta_{gg}^0}{C_{21}^2} \quad (\text{convection scheme independent}).$$

$$= \frac{3}{2} \delta_{gg}^0 + (1 - \delta_{gg}^0)^2$$

$$= 1 + \frac{5}{2} \left(\frac{\omega_s}{N} \right) + \left(\frac{\omega_s}{N} \right)^2 + \left(\frac{\omega_s}{N} \right)^3 + 7 \left(\frac{\omega_s}{N} \right)^4 + 5 \cdot 8 \left(\frac{\omega_s}{N} \right)^5 + 13 \cdot 4 \left(\frac{\omega_s}{N} \right)^6 \\ + 58 \left(\frac{\omega_s}{N} \right)^7 + \dots$$

$$F_2^{(0)}(N, Q^2) = \omega_s(Q^2) \frac{\Gamma_{22}'}{\delta_{gg}^0} \left(e^{\delta_{gg}^0 t \ln Q^2/N} - 1 \right) F_2(N) + F_2^{(0)}(N)$$

$$\rightarrow F_2(x, Q^2) \sim x^{-0.3} \quad 0.01 \leq x \leq 10^{-5}$$

($10 \text{ GeV}^2 < Q^2 < 100 \text{ GeV}^2$).

Rough (correct) prediction for structure function inputs.

Strong prediction for relationship between inputs.

Renormalization Scheme Consistent Calculation of Structure Functions.

Leading order contains all physically important terms (in physical quantity) at lowest power of $\alpha_s(\mu^2)$.

i.e. lowest order for each power of N^{-1} .

Combines $\alpha_s(\mu^2)$ expansion and leading in $\ln(\frac{\mu}{\Lambda})$ expansion.

- Contains all terms which dominate in some region of parameter space.
- Guarantees consistency with given renormalization scheme.

LO expression renormalization scheme independent by definition.

Only LO expressions exist so far (I don't know what next to leading is for $\bar{t} \bar{t} \rightarrow t \bar{t} \gamma \gamma$ & $t \bar{t} \rightarrow t \bar{t}$ and).

See point out.

Expressions work very well. Give more predictive power at small x than NLO in $\ln(x)$ approach, and are more constrained at small x . They give a better global fit. (particularly at small x).

Castor doubts on determination of $\Delta \bar{s}(M_B^2)$ using NLO in $\ln(x)$ fits to $F_2(x, Q^2)$.

May also be tested by measurement of $F_L(x, Q^2)$. (Predicted from fit to F_2).
Different to NLO in $\ln(x)$ approach.

Look at charm structure function.

Calculations in progress. Reasons to be optimistic.

Other more exclusive quantities. Calculations to be done.

(When NLO in $\ln(x)$ expressions for coefficient functions and anomalous dimensions become available (soon?) wish to calculate real NLO renormalization scheme consistent structure functions.

Real measurement of $\Delta \bar{s}(M_B^2) ?)$

Conclusions

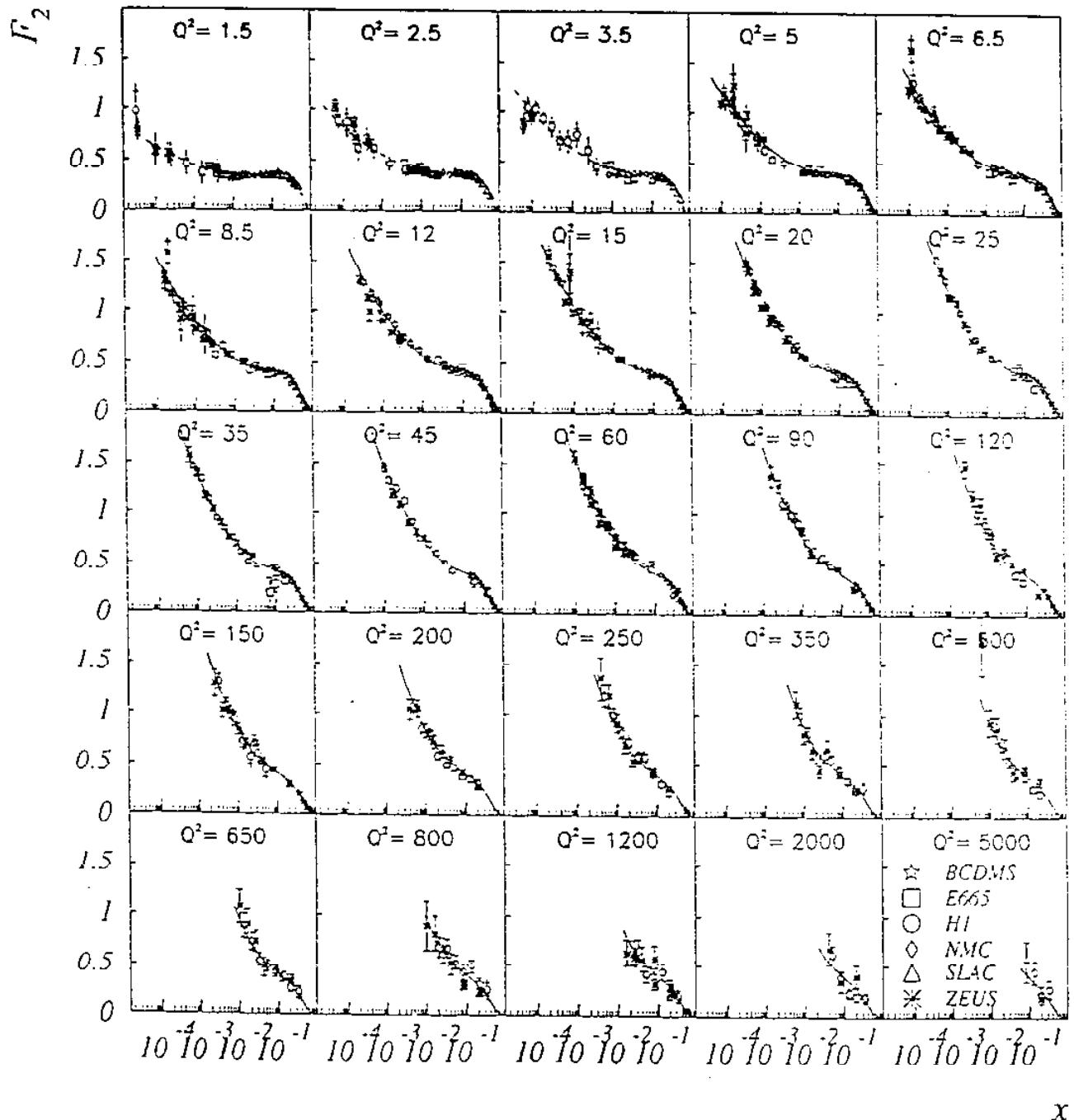
Wolt ordered calculations of structure functions are not factorization scale dependent.

Inputs for structure functions estimated by looking at variation of evolution.

Correct calculation includes looking in t_3) terms and leading $ts(0^2)$ terms automatically.

All the above works very well comparing current calculations with enceint data.

More calculations needed, and more, and varied measurements (especially of F_{2e}, α^2) needed, to truly determine correct approach for hadron interactions.



x

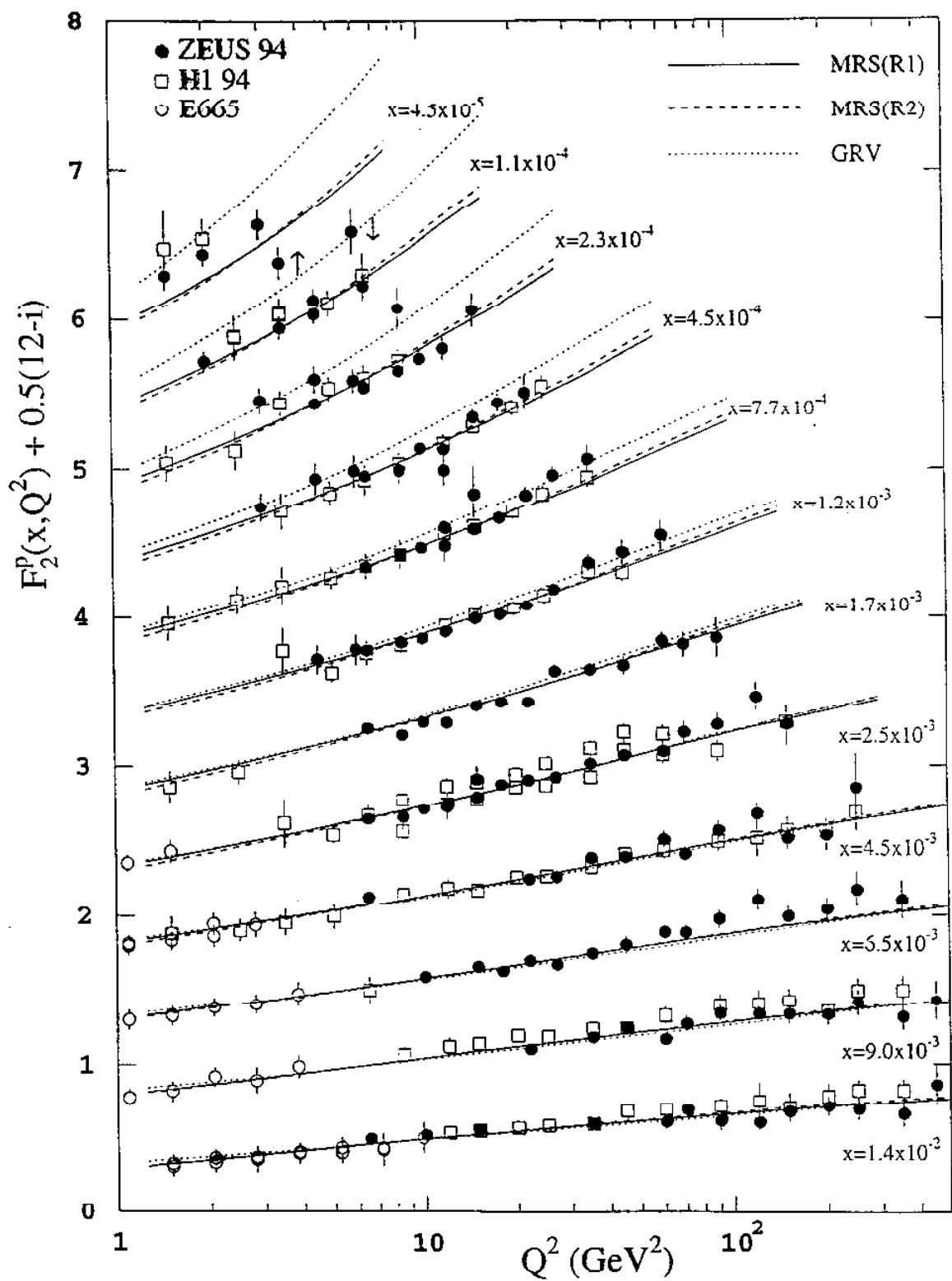


Fig. 2

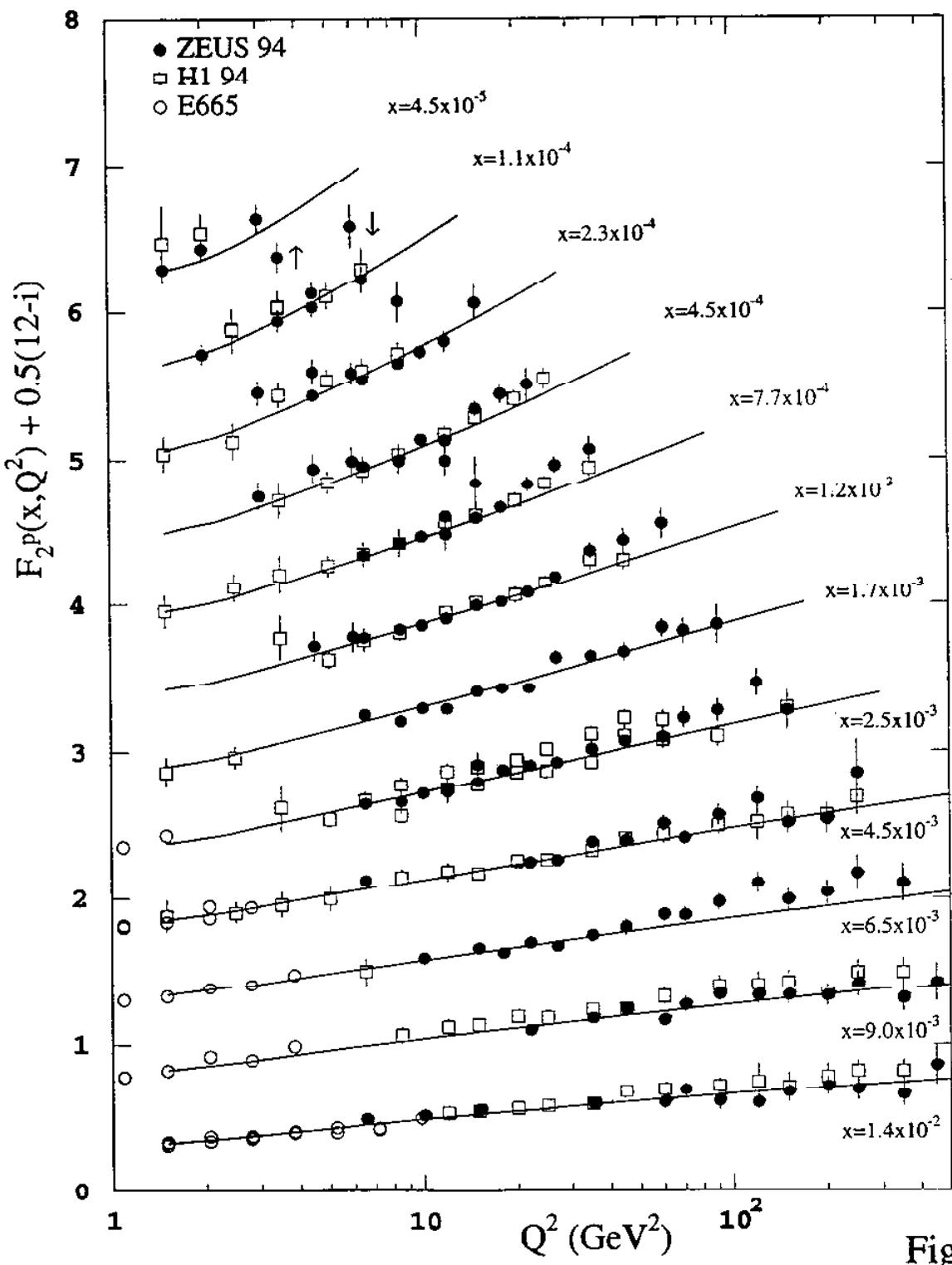


Fig. 2

$$\begin{aligned}
F_{L,RSC,0}(N, Q^2) = & \frac{\alpha_s(Q_0^2)}{2\pi} \left[\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\Gamma^{0,l,+}(N)-1} \exp \left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\Gamma_{LL}^0(N, \alpha_s(q^2))}{b_0 \alpha_s^2(q^2)} d\alpha_s(q^2) \right] \times \right. \\
& \left(\hat{F}_L^{0,l,+}(N) + \left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27} \right) F_2(N) \right) (\exp[\ln(Q_0^2/A_{LL}) \Gamma_{LL}^0(N, \alpha_s(Q_0^2))] - 1) \right. \\
& \left. \left. + \hat{F}_L^{0,l,-}(N) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\Gamma^{0,l,-}(N)-1} \right] \right] \tag{2.13}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d F_2(N, Q^2)}{d \ln Q^2} \right)_{RSC,0} = & \alpha_s(Q_0^2) \left[e^-(N) \Gamma^{0,l,-}(N) \hat{F}_L^{0,l,-}(N) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\Gamma^{0,l,-}(N)-1} \right. \\
& + \left(e^+(N) \Gamma^{0,l,+}(N) \hat{F}_L^{0,l,+}(N) - \Gamma_{2,L}^{1,0}(N) \left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27} \right) F_2(N) \right) \right. \\
& \left. + \Gamma_{2,L}^1(N, \alpha_s(Q_0^2)) \left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27} \right) F_2(N) \right) \exp[\ln(Q_0^2/A_{LL}) \Gamma_{LL}^0(N, \alpha_s(Q_0^2))] \right) \times \\
& \left. \exp \left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\Gamma_{LL}^0(N, \alpha_s(q^2))}{b_0 \alpha_s^2(q^2)} d\alpha_s(q^2) \right] \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\Gamma^{0,l,+}(N)-1} \right] \tag{2.14}
\end{aligned}$$

$$\begin{aligned}
F_{2,RSC,0}(N, Q_0^2) = & F_2(N) \\
& + \alpha_s(Q_0^2) \frac{\Gamma_{2,L}^1(N, \alpha_s(Q_0^2))}{\Gamma_{LL}^0(N, \alpha_s(Q_0^2))} \left(\hat{F}_L(N) - \frac{(36-8N_f)}{27} F_2(N) \right) (\exp[\ln(Q_0^2/A_{LL}) \Gamma_{LL}^0(N, \alpha_s(Q_0^2))] - 1) \\
& + \ln(Q_0^2/A_{LL}) \alpha_s(Q_0^2) \left(e^+(N) \Gamma^{0,l,+}(N) \hat{F}_L^{0,l,+}(N, Q_0^2) + e^-(N) \Gamma^{0,l,-}(N) \hat{F}_L^{0,l,-}(N, Q_0^2) \right. \\
& \left. - \Gamma_{2,L}^{1,0}(N) \left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27} \right) F_2(N) \right) \right) \tag{2.15}
\end{aligned}$$

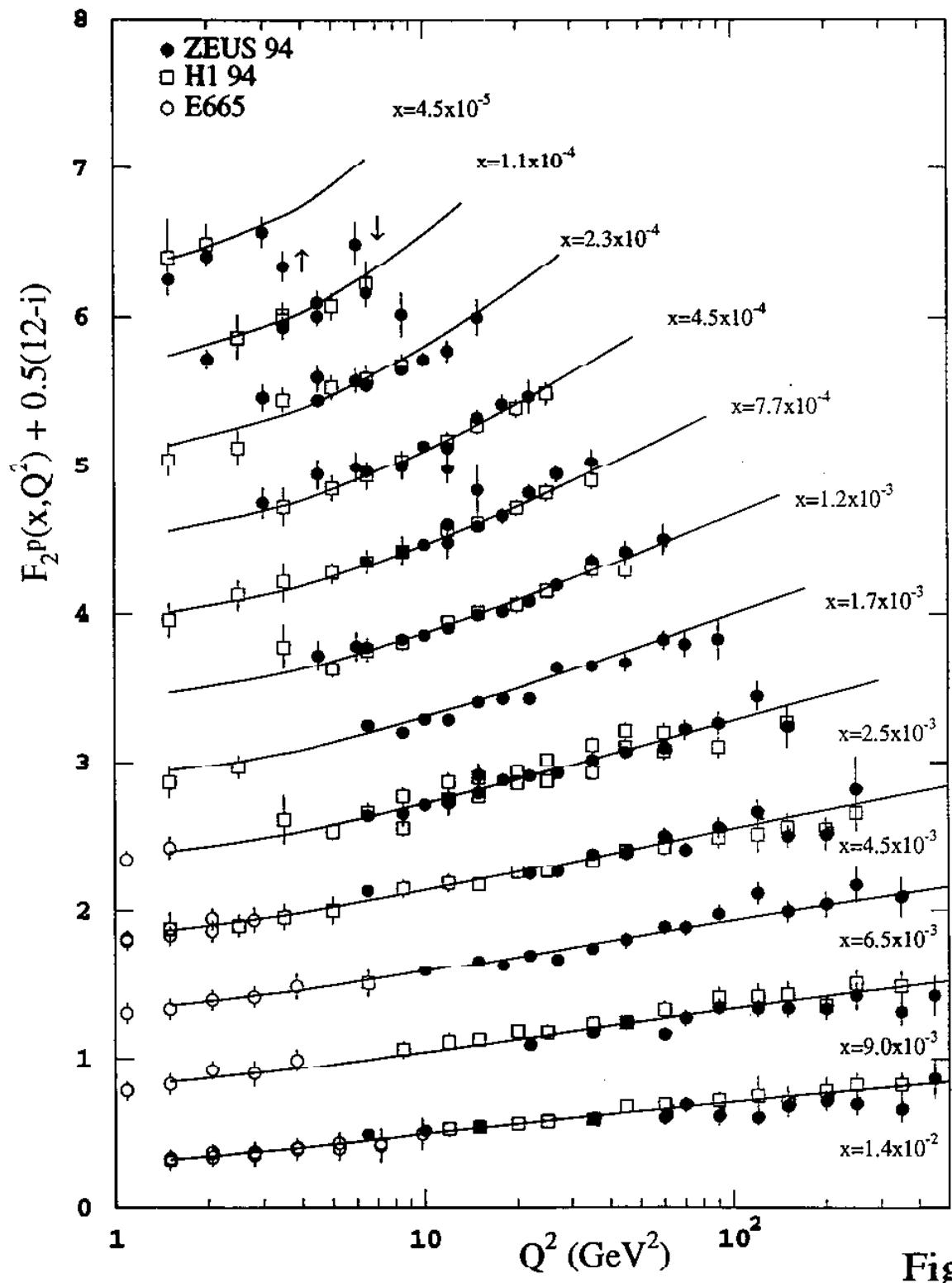


Fig. 1

Table 1

Comparison of quality of fits using the full leading-order (including leading- $\ln(1/x)$ terms) renormalization-scheme-consistent expression, LO(x), and the two-loop fits MRSR₁, MRSR₂, NLO₁ and NLO₂. For the LO(x) fit the H1 data chooses a normalization of 1.00, the ZEUS data of 1.015, and the BCDMS data of 0.975. The CCFR data is fixed at a normalization of 0.95, and the rest is fixed at 1.00. Similarly, for the NLO₁ fit the H1 data is fixed at a normalization of 0.985, the ZEUS chooses a normalization of 0.99, and the BCDMS data of 0.975. Again the CCFR data is fixed at a normalization of 0.95, and the rest fixed at 1.00. Also, for the NLO₂ fit the H1 data is fixed at a normalization of 0.985, the ZEUS chooses a normalization of 0.985, and the BCDMS data of 0.97. Again the CCFR data is fixed at a normalization of 0.95, and the rest fixed at 1.00. In the R₁ and R₂ fits the BCDMS data has a fixed normalization of 0.98, the CCFR data of 0.95 and the rest of 1.00.

Experiment	data points	χ^2				
		LO(x)	NLO ₁	NLO ₂	R ₁	R ₂
H1 $F_2^{\epsilon p}$	193	123	145	145	158	149
ZEUS $F_2^{\epsilon p}$	204	253	281	296	326	308
BCDMS $F_2^{\mu p}$	174	181	218	192	265	320
NMC $F_2^{\mu p}$	120	122	131	148	163	135
NMC $F_2^{\mu d}$	129	114	107	125	134	99
NMC $F_2^{\mu n}/F_2^{\mu p}$	85	142	137	138	136	132
E665 $F_2^{\mu p}$	53	63	63	63	62	63
CCFR $F_2^{\nu N}$	66	59	48	40	41	56
CCFR $F_2^{\nu N}$	66	48	39	36	51	47

1099 1105 1169 1184

Table 2

Comparison of quality of fits using the full leading-order (including leading- $\ln(1/x)$ terms) renormalization-scheme-consistent expression, LO(x), and the two-loop fits NLO₁ and NLO₂. The fits are identical to above, but the data are presented in terms of whether x is less than 0.1 or not.

	data points	χ^2		
		LO(x)	NLO ₁	NLO ₂
$x \geq 0.1$	551	622	615	595
$x < 0.1$	548	483	554	580
total	1099	1105	1169	1184

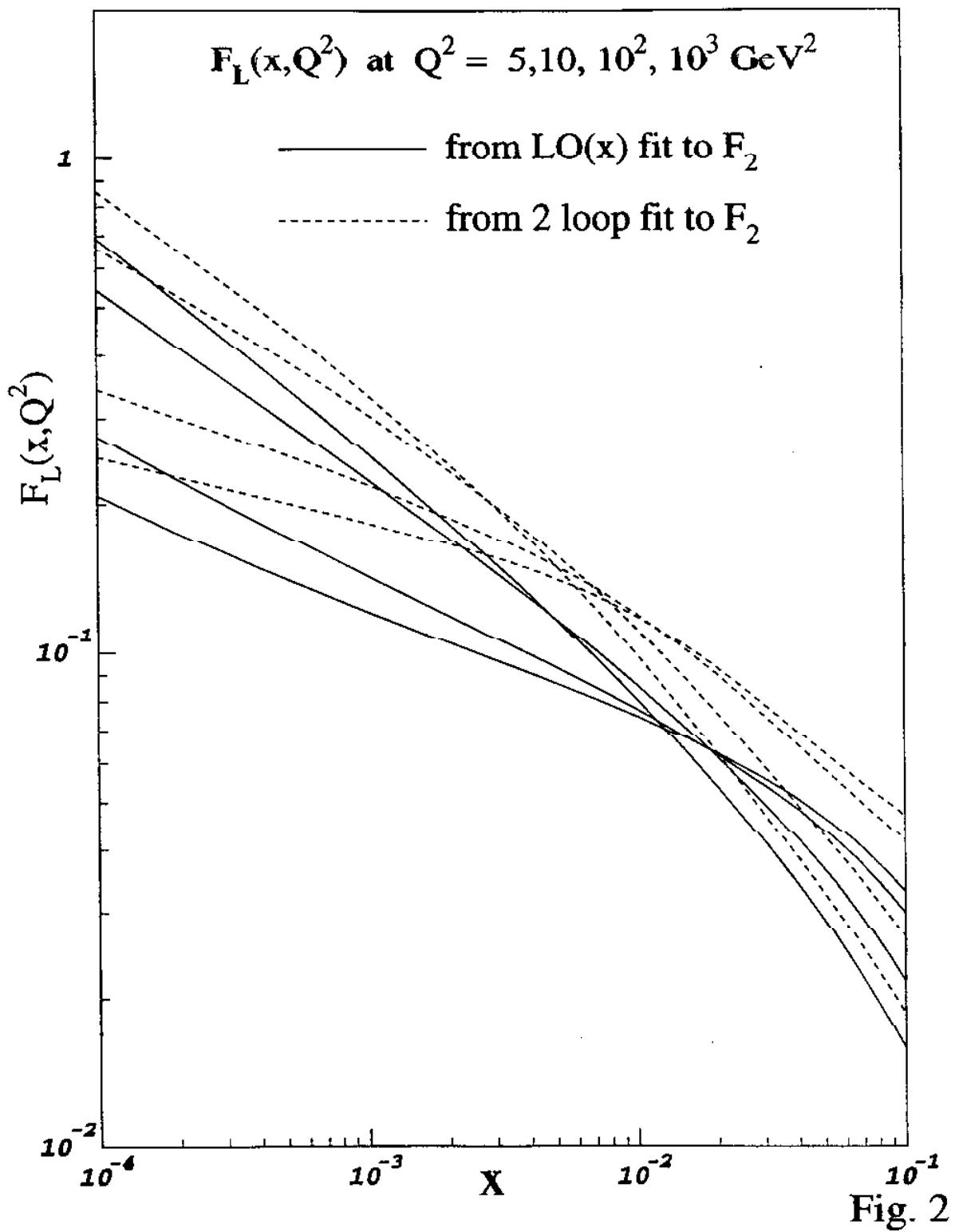
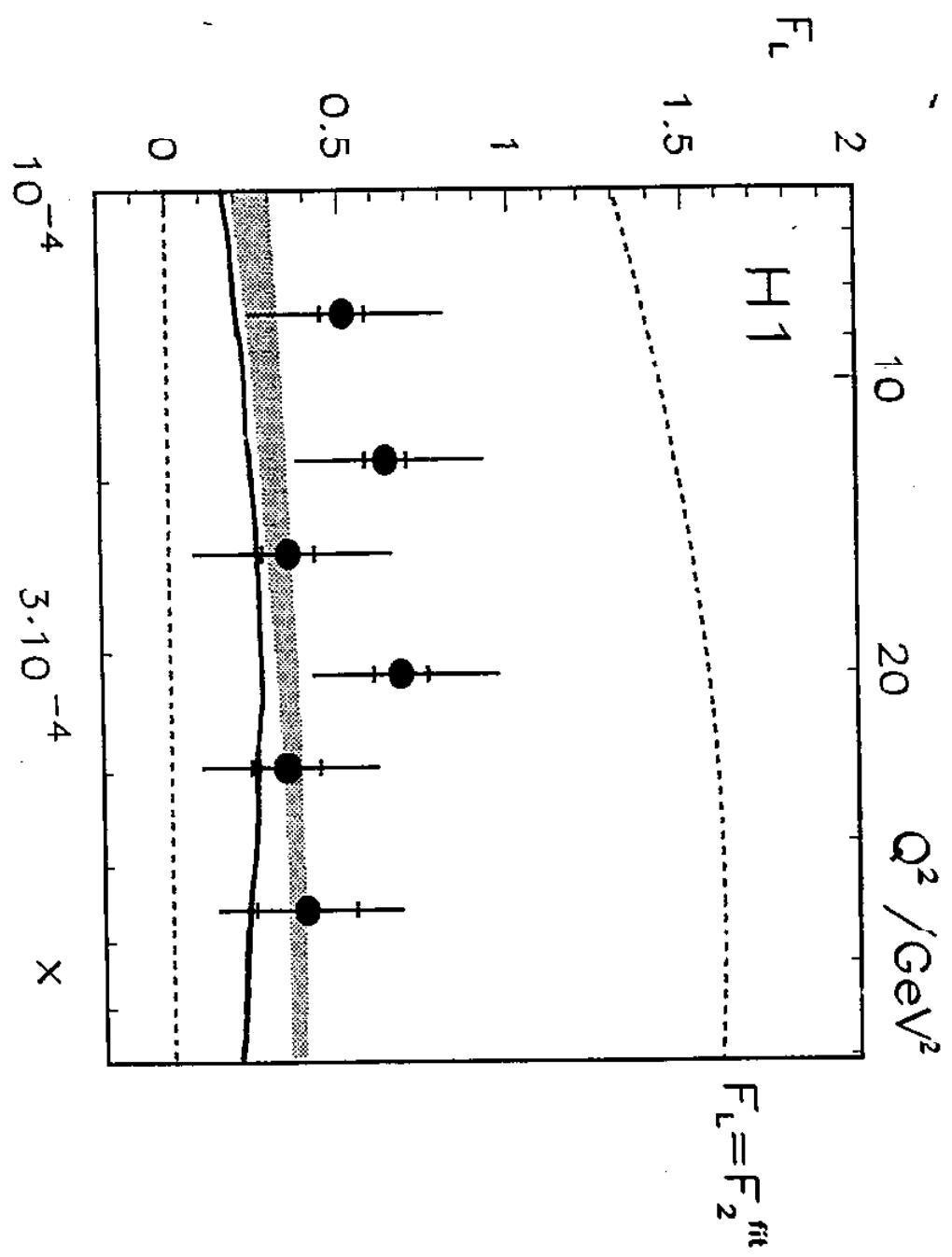
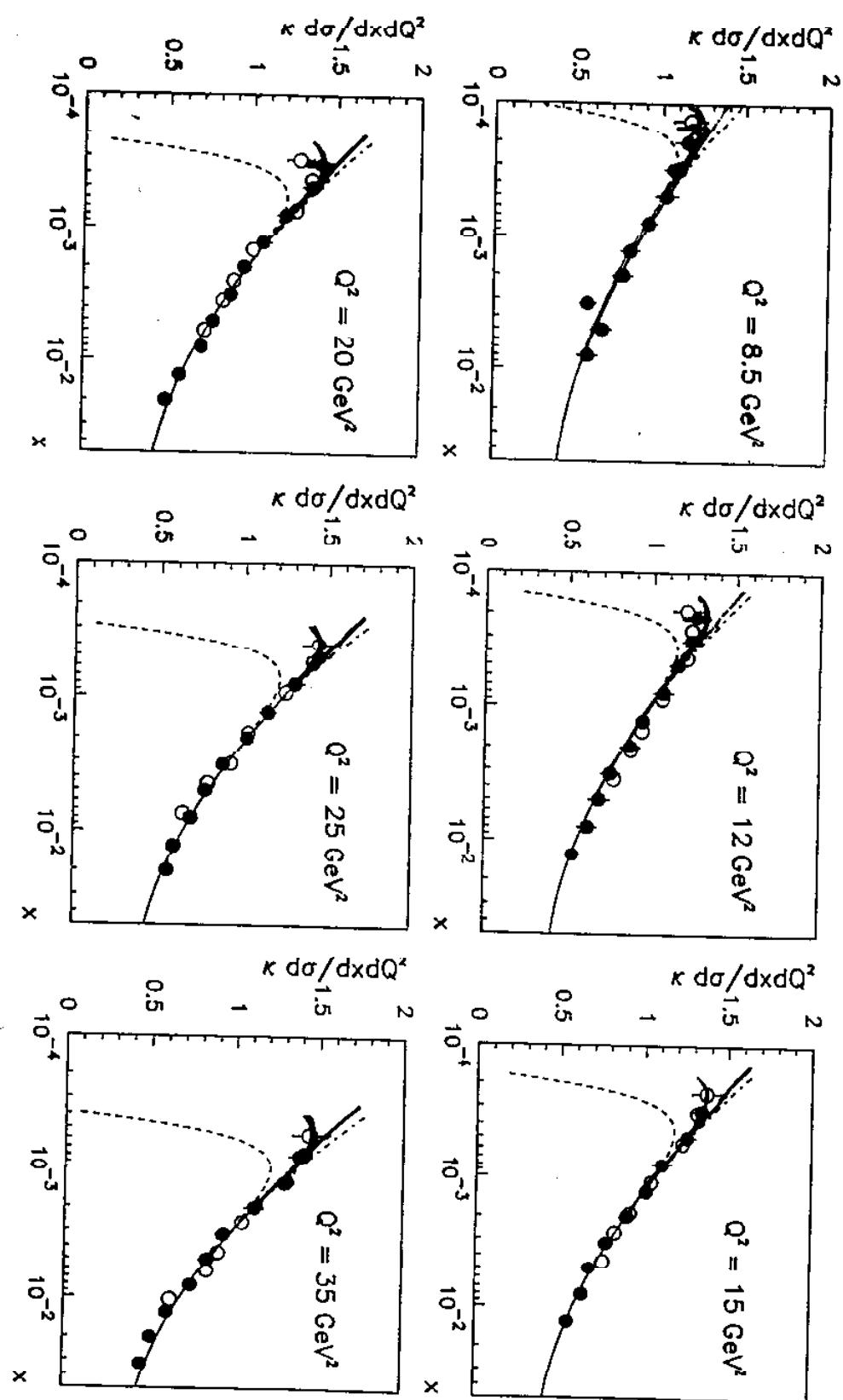


Fig. 2



--- $F_L = 0$
 --- $F_L = \text{LORE} F_L$
 \bullet Z_veto



I

